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Ordering problem in geometrical quantization

Dehai Bao and Zhong-yuan Zhu†

CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China,
and Institute of Theoretical Physics, Academia Sinica, PO Box 2735, Beijing 100080,
People's Republic of China

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Abstract. The operator ordering problem is investigated in the framework of geometrical quantization in the Schrödinger coordinate representation. It is shown that, for quantization of polarization-preserving classical observables $pf(q)$, correct Hermitian quantum operators are derived; for quantization of polarization-unpreserving classical observables $p^2f(q)$, the BKS kernel method actually defines an ordering rule which is not Weyl's rule, but is the GCT-invariant rule proposed by DeWitt.

The problem of operator ordering ambiguity in quantum mechanics is a long-standing one and has been examined by many authors both in the canonical quantization formalism and the Feynman path integral quantization formalism (see [1] for a survey of work prior to 1959, also [2–15]). It has been proved that the ordering ambiguity exists in both procedures. In order to get an unique quantum system corresponding to a given classical one, we have to choose some ordering rule (or equivalently, some definition of discrete form for action in the path integral formalism). Different ordering rules usually lead to non-equivalent quantum operators. It is not known which one should be adopted, as each of them has certain advantages, but none of them is completely satisfactory [1, 14–15].

Geometrical quantization is a comparatively new method of quantization developed in 1970s [16–20]. In geometrical quantization formalism, there exists a well-defined operation on states in Hilbert space corresponding to a classical observable and as a result, geometrical quantization must be free from the problem of ambiguity. In other words, a certain ordering rule must have been defined by the formalism itself. It is of the interest to see what this rule is like.

As the ordering problem is what we address this paper, it is convenient to work in one dimensional situations. The configuration space is $Q = \{q\}$ and the classical phase space is $M = T^*Q$. Therefore, the symplectic structure is simply $\omega = dp \wedge dq$ and the symplectic potential $\theta = p dq$.

In order to express our results in familiar forms of usual quantum mechanics, we will discuss them in the Schrödinger coordinate representation. In other words, the polarization structure needed is chosen to be F spanned by $X = \partial/\partial p$. (Since M is contractible, all the additional structure needed for quantization exists [19].)

Under the chosen polarization F , Hilbert space consists of polarized sections of $L \otimes \sqrt{\wedge} F$ with forms as $s = \psi(q)s_0 \otimes \nu$, where $\psi(q)$ is smooth complex-valued function

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over R , s_0 is the unit section of prequantum bundle $L = M \otimes C$ and ν is the section of the half-form bundle for F .

For any classical observable, geometrical quantization gives a well-defined operation on states of Hilbert space.

1. Let us first consider the quantization of classical observables which preserve the polarization F . These kinds of quantities can be generally expressed as follows:

$$g(p, q) = pf(q) + V(q). \quad (1)$$

The induced Hamiltonian vector field is,

$$X_g = f(q) \frac{\partial}{\partial q} - [pf'(q) + V'(q)] \frac{\partial}{\partial p}. \quad (2)$$

Therefore,

$$\left[X_g, \frac{\partial}{\partial p} \right] = f'(q) \frac{\partial}{\partial p}. \quad (3)$$

As a result, the operation on states of Hilbert space induced from g is as follows [19-20]:

$$O(g)[\psi s_0 \otimes \nu] = \left[(-i\hbar \nabla_{X_g} + g) - \frac{i\hbar}{2} f'(q) \right] (\psi s_0 \otimes \nu) \quad (4)$$

where $\nabla_{X_g} = X_g - (i/\hbar)\theta(X_g)$.

Substituting (2) into (4) directly gives,

$$O(g)[\psi s_0 \otimes \nu] = \left[-i\hbar f(q) \frac{\partial}{\partial q} + V(q) - \frac{i\hbar}{2} f'(q) \right] (\psi s_0 \otimes \nu). \quad (5)$$

In the usual language of quantum mechanics, the above result means,

$$\begin{aligned} O(pf(q) + V(q)) &= -i\hbar f(q) \frac{\partial}{\partial q} - \frac{i\hbar}{2} f'(q) + V(q) \\ &= f(Q)P - \frac{i\hbar}{2} f'(Q) + V(Q) \end{aligned} \quad (6)$$

where $P = O(p) = -i\hbar(\partial/\partial q)$, $Q = O(q) = q$.

Obviously, result (6) is just the correct operator of $pf(q) + V(q)$, which is unique when Hermiticity is required [15].

2. Now let us consider the quantization of classical quantities $g(p, q) = p^2 f(q)$ which do not preserve polarization F . The Hamiltonian vector field induced from g is as follows:

$$X_g = 2pf(q) \frac{\partial}{\partial q} - p^2 f'(q) \frac{\partial}{\partial p}. \quad (7)$$

According to the BKS method, the operators of classical quantities which do not preserve polarization are given by the following formula [19-20]:

$$O(g)[\psi s_0 \otimes \nu] = i\hbar \frac{d}{dt} \psi_t(s_0 \otimes \nu)|_{t=0} \quad (8)$$

where $\psi_t(q)$ is defined by:

$$\psi_t(q) = (i\hbar)^{-1/2} \int_R dp \cdot \exp \left[\frac{i}{\hbar} \int_0^t (\theta X_g - g) \circ \phi_g^{-s} ds \right] \times [\det \omega(X_q, \phi_g^t X_q)]^{1/2} \cdot \psi(\phi_g^t q) \tag{9}$$

and ϕ_g^t is the canonical transformation induced by g .

Straightforward calculation with the aid of (7) gives

$$\int_0^t [\theta(X_g) - g] \circ \phi_g^{-s} ds = \xi_0^t g \circ \phi_g^{-s} ds = tg. \tag{10}$$

Substituting (10) into (9) and changing the variable of integration p to $u = tp$, we get:

$$\psi_t(q) = (i\hbar)^{-1/2} \int_R \frac{du}{t} \exp \left[\frac{i}{\hbar} \frac{f(q)}{t} u^2 \right] [\det \omega(X_q, \phi_g^t X_q)]^{1/2} \psi(\phi_g^t q). \tag{11}$$

In order to evaluate (8) with (11), let us make a Taylor expansion for $\phi_g^t q = q \circ \phi_g^{-t}$ as follows:

$$\phi_g^t q = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} q^{(n)} t^n. \tag{12}$$

According to the following motion equations induced from g ,

$$\dot{q} = 2pf(q) \quad \dot{p} = -p^2 f'(q). \tag{13}$$

It is easy to see that $q^{(n)}$ can be expressed as $q^{(n)} = \beta_n(q) p^n$ where $\beta_n(q)$ is a function of q only. Therefore,

$$\phi_g^t q = \sum_0^{\infty} \frac{(-1)^n}{n!} \beta_n(q) p^n t^n \tag{14}$$

and

$$\omega(X_q, \phi_g^t X_q) = \{q, \phi_g^t q\}_{PB} = \sum_0^{\infty} \frac{(-1)^n}{n!} t^n \{q, \beta_n(q) p^n\}_{PB} = \sum_1^{\infty} \frac{(-1)^{n-1}}{(n-1)!} \beta_n(q) p^{n-1} t^n. \tag{15}$$

Substituting (14) and (15) into (11) and then into (8), we get,

$$Og[\psi_{s_0} \otimes \nu] = (i\hbar) \left(\frac{2f(q)}{i\hbar} \right)^{1/2} \lim_{t \rightarrow 0} \frac{d}{dt} \int_R du t^{-1/2} \exp \left(\frac{i}{\hbar} \frac{f(q)}{t} u^2 \right) \times \left[\sum_1^{\infty} \frac{(-1)^{n-1}}{(n-1)!} \frac{\beta_n}{2f(q)} u^{n-1} \right]^{1/2} \psi \left(\sum_0^{\infty} \frac{(-1)^n}{n!} \beta_n u^n \right) [s_0 \otimes \nu]. \tag{16}$$

Now, taking into account the fact that, for real variables s and $a > 0$,

$$\lim_{t \rightarrow 0^+} t^{-1/2} \exp(ias^2/t) = (\pi/a)^{1/2} e^{-i\pi/4} \delta(s) \tag{17}$$

where $\delta(s)$ is the Dirac distribution, and

$$\left(\frac{\partial^2}{\partial s^2} + 4ia \frac{\partial}{\partial t} \right) t^{-1/2} \exp(ias^2/t) = 0 \tag{18}$$

we get the following result from (16):

$$\begin{aligned}
 O(g)[\psi s_0 \otimes \nu] &= \frac{(i\hbar)^2}{4f(q)} \int_R du \left[\sum_1^\infty \frac{(-1)^{n-1} \beta_n(q)}{(n-1)! 2f(q)} u^{n-1} \right]^{1/2} \\
 &\quad \times \psi \left(\sum_0^\infty \frac{(-1)^n}{n!} \beta_n u^n \right) \frac{d^2}{du^2} (\delta(u)) [s_0 \otimes \nu]. \tag{19}
 \end{aligned}$$

According to the property of the Dirac δ -function,

$$\int_{-\infty}^{+\infty} ds F(s) \frac{d^2}{ds^2} \delta(s) = \frac{d^2}{ds^2} F(s)|_{s=0}. \tag{20}$$

Equation (19) can be easily evaluated as follows:

$$\begin{aligned}
 O(g)[\psi s_0 \otimes \nu] &= \frac{(i\hbar)^2}{4f(q)} \left\{ 2f\beta_1 \left(\frac{\beta_1}{2f} \right)^{3/2} \psi'' + 2\beta_2 \left(\frac{\beta_1}{2f} \right)^{1/2} \psi' \right. \\
 &\quad \left. + \left[\beta_3 \left(\frac{\beta_1}{2f} \right)^{-1/2} - \frac{\beta_2^2}{16f^2} \left(\frac{\beta_1}{2f} \right)^{-3/2} \right] \psi \right\} [s_0 \otimes \nu]. \tag{21}
 \end{aligned}$$

Now, according to (13), it is easy to derive that,

$$\beta_1 = 2f \quad \beta_2 = 2f'f \quad \beta_3 = 4f''f^2. \tag{22}$$

Substituting (22) into (21), we finally get,

$$O(g)[\psi s_0 \otimes \nu] = (i\hbar)^2 \left\{ f(q)\psi''(q) + f'(q)\psi'(q) + \left[\frac{f''(q)}{4} - \frac{f'(q)^2}{16f(q)} \right] \psi(q) \right\} [s_0 \otimes \nu]. \tag{23}$$

In the usual language of quantum mechanics, the above result means,

$$O(p^2f(q)) = f(Q)P^2 - i\hbar f'(Q)P + (i\hbar)^2 \left[\frac{f''(Q)}{4} - \frac{f'(Q)^2}{16f(Q)} \right]. \tag{24}$$

Referring to [2, 14, 15], we find that the above operator is exactly ordered by the GCT (general coordinates transformation)-invariant ordering rule proposed by DeWitt and clarified by Dehai Bao *et al.* (One should be careful that (24) is derived for those $f(q) > 0$ since, otherwise (17) cannot be used.) This GCT-invariant ordered operator is Hermitian as physically required. However, it differs from, at \hbar^2 level, the commonly adopted Weyl rule which gives the operator form of $p^2f(q)$ as follows [21]:

$$\begin{aligned}
 O_{\text{Weyl}}(p^2f(q)) &= \frac{1}{4}(P^2f(Q) + 2Pf(Q)P + f(Q)P^2) \\
 &= f(Q)P^2 - i\hbar f'(Q)P + (i\hbar)^2 \frac{f''(Q)}{4}. \tag{25}
 \end{aligned}$$

In the authors' opinion, it seems that BKS method bears GCT-invariant ordered operator for $p^2f(q)$ is a not too surprising result, since geometric quantization is essentially a geometric, coordinate-free construction for Hilbert space and observables of the underlying quantum theory. In fact, we might argue that, for more general cases (e.g. $p^n f(q)$), quantum operators given by geometric quantization must also be ordered by the GCT-invariant ordering rule, because of its coordinate-free nature. Unfortunately,

a general concrete verification of this argument seems far from straightforward and is also mathematically complicated. (For example, there lack formulae analogous to (17), (18) in the general case.) A further investigation on this issue and other related problems concerning the vks-method is underway.

As a conclusion, we have discussed the ordering problem in framework of geometrical quantization. Our results are that, for polarization-preserving classical observables $pf(q)$, geometrical quantization method gives their operators as in (6) which is correct and unique when Hermiticity is required; for classical quantities $p^2f(q)$ with $f(q) > 0$, the vks kernel method gives their operators as in (24) which are actually ordered by the GCT-invariant ordering rule. In other words, we may make the following two remarks: (a) unlike canonical quantization and path integral quantization, geometrical quantization itself has defined an ordering rule and thus is free from the problem of ordering ambiguity; (b) the ordering rule defined by geometrical quantization is the GCT-invariant rule but not the commonly adopted Weyl rule in canonical quantization formalism (or equivalently, mid-point-rule in the path integral formalism). This statement is verified explicitly for a simple but non-trivial example as in (24), (25) and is argued to hold true generally.

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